

# CS 4810: Homework 1

due 09/05 11:59pm

(your name + netid)

Collaborators: (names and netids)

Each problem is worth 20 points.

## Problem 0

Submit the student information form (see link on piazza).

## Problem 1

Show that every  $n$ -bit boolean function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$  can be represented by a circuit of depth at most  $O(n)$ .

## Problem 2

Let  $\text{NOR} : \{0, 1\}^2 \rightarrow \{0, 1\}$  be the not-or operation, so that  $\text{NOR}(x, y) = 0$  if  $x = 1$  or  $y = 1$ , and  $\text{NOR}(x, y) = 1$  if  $x = y = 0$ .

Show that every circuit can then also be implemented using only NOR gates (instead of NOT, AND, OR). Show also that this does not affect the asymptotic size or depth of the circuit. (Concretely, show that for every circuit  $C$ , there exists a circuit  $C'$  using only NOR gates, such that  $C$  and  $C'$  compute the same function and have the same size and depth up to constant factors, i.e.,  $\text{size}(C') = O(1) \cdot \text{size}(C)$  and  $\text{depth}(C') = O(1) \cdot \text{depth}(C)$ .)

You may assume that in addition to the function inputs, you have access to constants, i.e., wires that are always 0 or always 1.

### Problem 3

The function PARITY:  $\{0, 1\}^n \rightarrow \{0, 1\}$  returns 1 if and only if an odd number of its inputs are 1.

Show that PARITY can be represented by a circuit of size  $O(n)$  and depth  $O(\log n)$ .

### Problem 4

A *labelled tree* with  $n$  labels is a tree – that is, a connected graph without cycles – whose  $n$  vertices are the integers from 1 up to  $n$ .

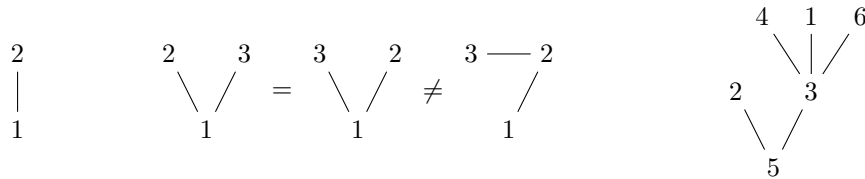


Figure 1: Some drawings of labelled trees, not all distinct.

Show that for a given  $n$ , there are no more than  $n^n$  distinct labelled trees.

### Problem 5

The function MAJORITY:  $\{0, 1\}^n \rightarrow \{0, 1\}$  returns 1 if *at least*  $n/2$  of its inputs are 1, and 0 if *fewer than*  $n/2$  of them are.

Show that MAJORITY can be implemented with a circuit of size  $O(n^2)$ .