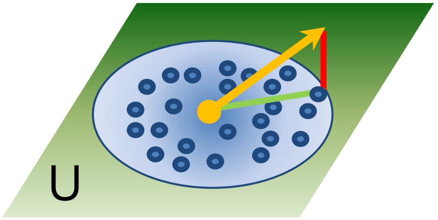
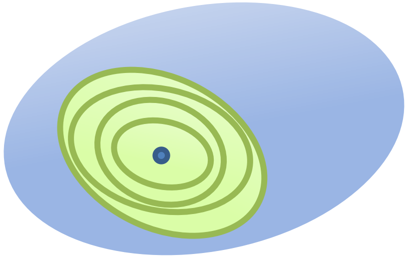
Subexponential Algorithms for Unique Games and Related Problems

Sanjeev Arora
Princeton University & CCI

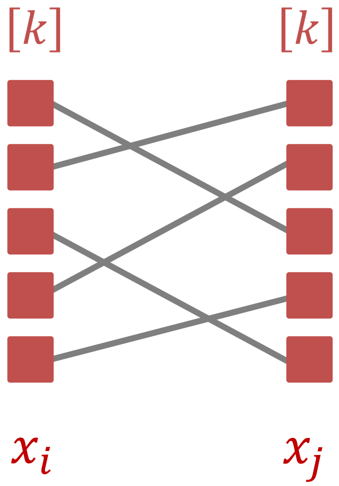
Boaz Barak
MSR New England & Princeton

David Steurer
MSR New England





UC Berkeley Theory Seminar, January 2011



UNIQUE GAMES

Input: list of constraints of form $x_i - x_j = c \bmod k$

Goal: satisfy as many constraints as possible

Unique Games

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Goal: satisfy as many constraints as possible

Unique Games Conjecture (UGC) [Khot'02]

For every $\varepsilon > 0$, the following is NP-hard:

$\mathrm{UG}(\varepsilon)$

Input: **Unique Games** instance with $k \ll \log n$ (say)

Goal: Distinguish two cases

YES: more than $1 - \varepsilon$ of constraints satisfiable

NO: less than ε of constraints satisfiable

Implications of UGC

For many basic optimization problems,
it is NP-hard to beat current algorithms
(based on simple LP or SDP relaxations)

Examples:

VERTEX COVER [Khot-Regev'03],
MAX CUT [Khot-Kindler-Mossel-O'Donnell'04,
Mossel-O'Donnell-Oleszkiewicz'05],
every **MAX CSP** [Raghavendra'08], ...

Implications of UGC

For many basic optimization problems,
it is NP-hard to beat current algorithms
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Unique Games Barrier

Example: $(\alpha_{\mathrm{GW}}+\varepsilon)$-approximation for **MAX CUT**
at least as hard as $\mathrm{UG}(\varepsilon')$

$\alpha_{\mathrm{GW}} = 0.878\ldots$
Goemans–Williamson bound for **MAX CUT**

UNIQUE GAMES is common barrier for improving current algorithms of many basic problems

Subexponential Algorithm for Unique Games

$\mathrm{UG}(\varepsilon)$ in time $\exp\left(n^{\varepsilon^{1/3}}\right)$

Time vs Approximation Trade-off

Input: **UNIQUE GAMES** instance with alphabet size k
such that $1 - \varepsilon$ of constraints are satisfiable,

Output: assignment satisfying $1 - C\sqrt{\varepsilon}$ of constraints

Time: $\exp\left(k\, n^{1/C^{2/3}}\right)$

Subexponential Algorithm for Unique Games

$\mathrm{UG}(\varepsilon)$ in time $\exp\left(n^{\varepsilon^{1/3}}\right)$

Consequences

NP-hardness reduction for $\mathrm{UG}(\varepsilon)$ must have blow-up $n^{1/\varepsilon^{1/3}}$ (*)
→ rules out certain classes of reductions for proving UGC

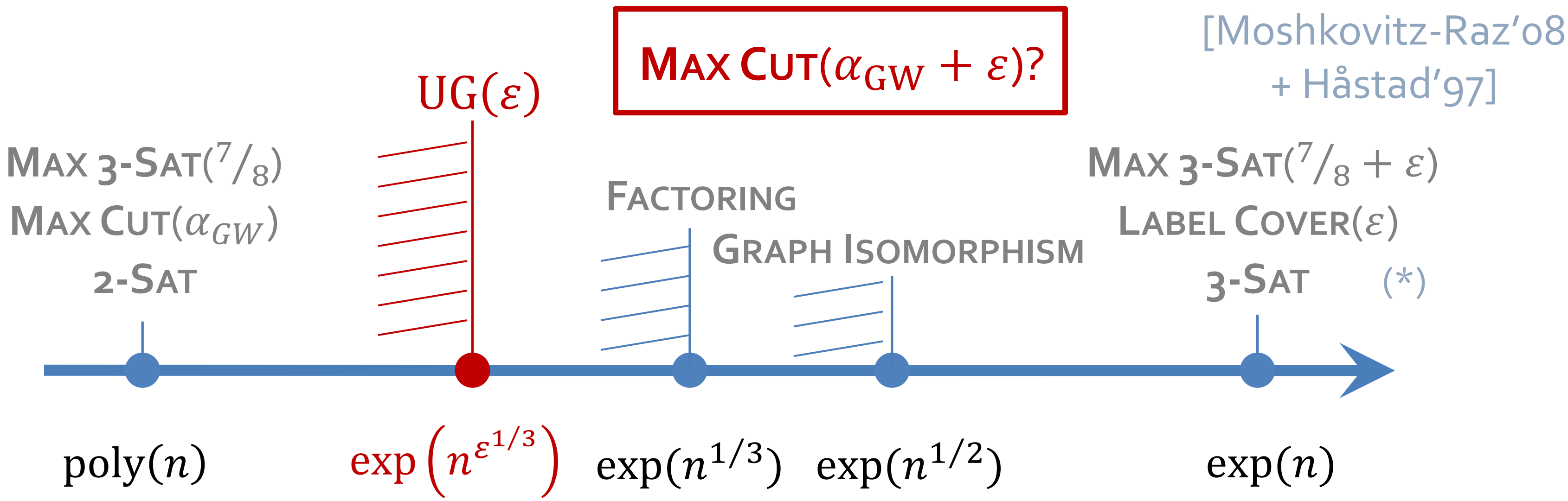
Analog of UGC with subconstant ε (say $\varepsilon = 1/\log\log n$) is false (*)
(*contrast*: subconstant hardness for LABEL COVER [Moshkovitz-Raz'08])

UGC-based hardness *does not rule out* subexponential algorithms,
→ *Possibility:* $\exp(n^{\varepsilon})$-time algorithm for **MAX CUT**$(\alpha_{\mathrm{GW}} + \varepsilon)$?

(*) assuming 3-SAT does not have subexponential algorithms, $\exp\left(n^{o(1)}\right)$

Subexponential Algorithm for Unique Games

UG(ε) in time $\exp\left(n^{\varepsilon^{1/3}}\right)$

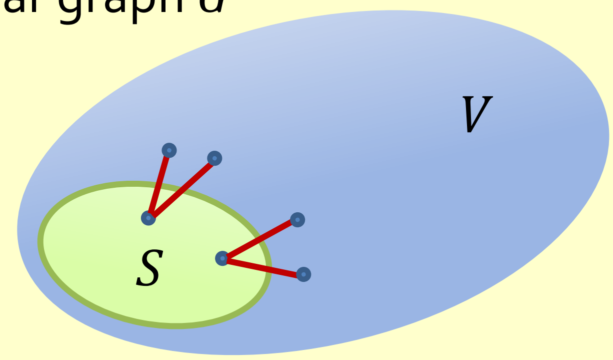


(*) assuming *Exponential Time Hypothesis* [Impagliazzo-Paturi-Zane'01]

(3-SAT has no $2^{o(n)}$ algorithm)

Interlude: Graph Expansion

d-regular graph G



expansion(S) = $\dfrac{\text{\# edges leaving } S}{d\ |S|}$

G = normalized adjacency matrix (*stochastic*)

x = normalized indicator vector

$$\text{expansion}(S) = \varepsilon \quad \Leftrightarrow \quad \langle x, Gx\rangle = 1 - \varepsilon$$

Subexponential Algorithm for Unique Games

$\mathrm{UG}(\varepsilon)$ in time $\exp\left(n^{\varepsilon^{1/3}}\right)$

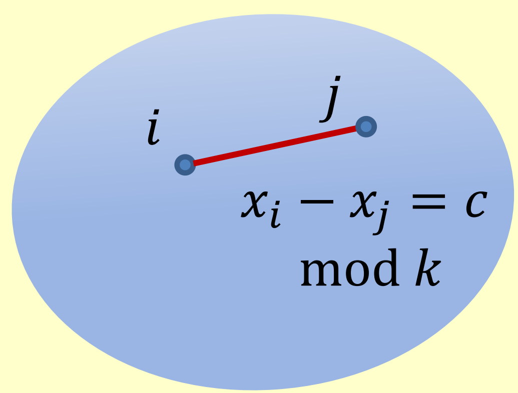
Easy graphs for **UNIQUE GAMES**

Expanding constraint graph [Arora-Khot-Kolla-S.-Tulsiani-Vishnoi'08]

$\mathrm{UG}(\varepsilon)$ in time $\mathrm{poly}(n)$ if eigenvalue gap $> 100\varepsilon$

Constraint Graph

variable → vertex
constraint → edge



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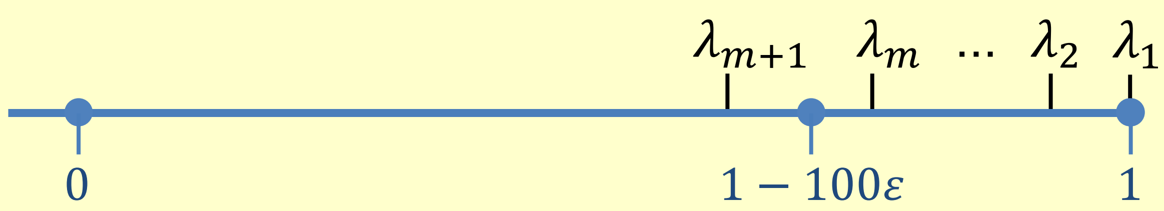
Constraint graph with few large eigenvalues [Kolla-Tulsiani'07, Kolla'10, *here*, Barak-Raghavendra-S.'11]

$\mathrm{UG}(\varepsilon)$ in time $\exp(m)$ if at most m eigenvalues $> 1 - 100\varepsilon$

quasi-expander

Graph Eigenvalues

normalized adjacency matrix



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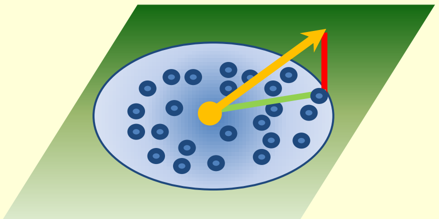
Subspace Enumeration [Kolla-Tulsiani'07]

exhaustive search over certain subspace

Question: Dimension of this subspace?



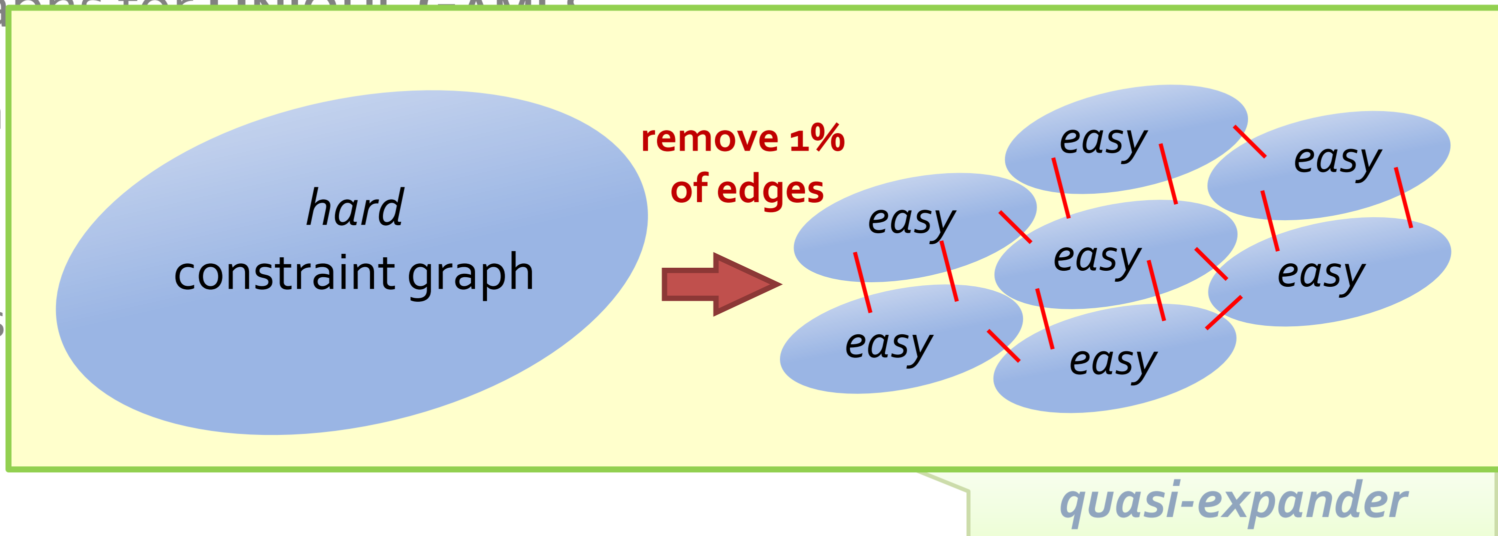
Subexponential Algorithm for Unique Games

$\mathrm{UG}(\varepsilon)$ in time $\exp\left(n^{\varepsilon^{1/3}}\right)$

Easy graphs for UNIQUE GAMES

Expa

Cons



Graph Decomposition

Approach [Trevisan'05, Arora-Impagliazzo-Matthews-S.'10]

By removing 1% of edges, decompose constraint graph into components for which $\mathrm{UG}(\varepsilon)$ is "easy"

Subexponential Algorithm for Unique Games

UG(ε) in time $\exp\left(n^{\varepsilon^{1/3}}\right)$

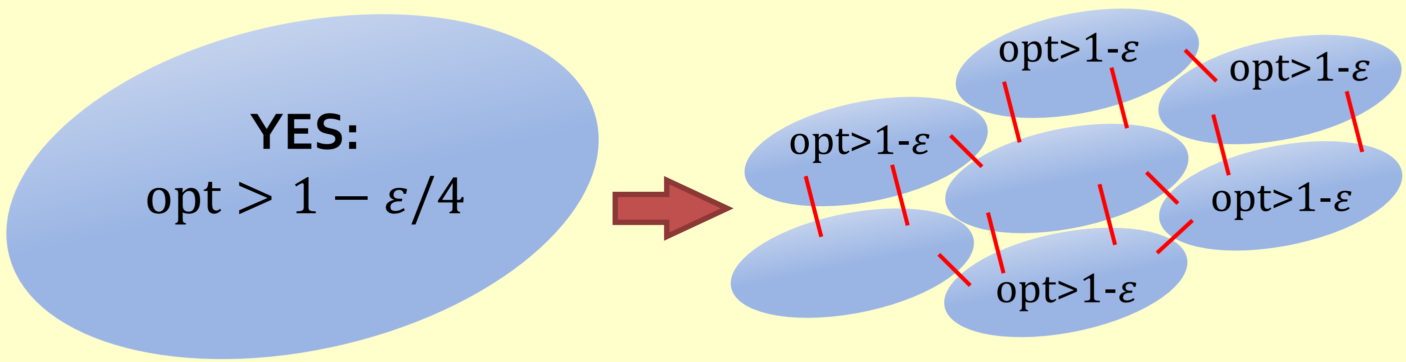
Easy graphs for UNIQUE GAMES

Expa

Cons

Graph D

Appr



UG(ε): Given UG instance, distinguish

YES: $\text{opt} > 1 - \varepsilon$

NO: $\text{opt} < \varepsilon$

Subexponential Algorithm for Unique Games

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Easy graphs for UNIQUE GAMES

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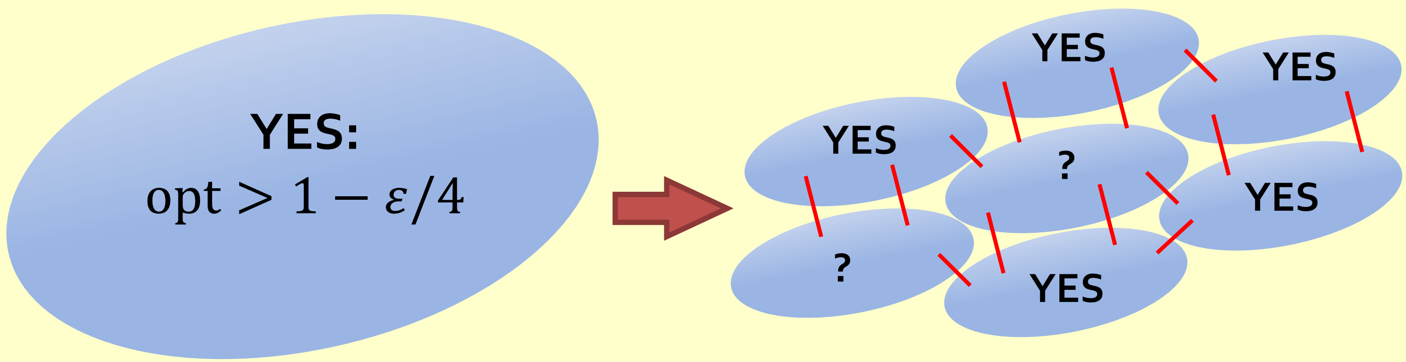

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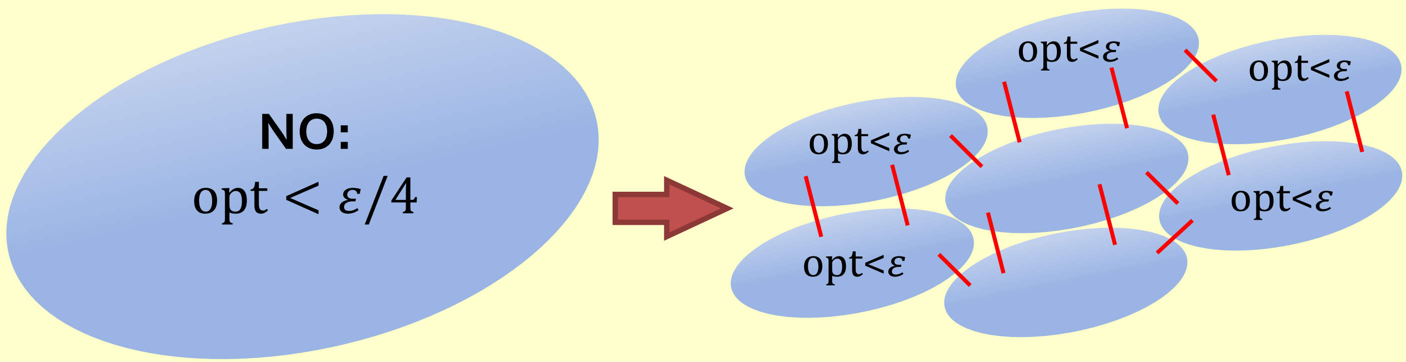

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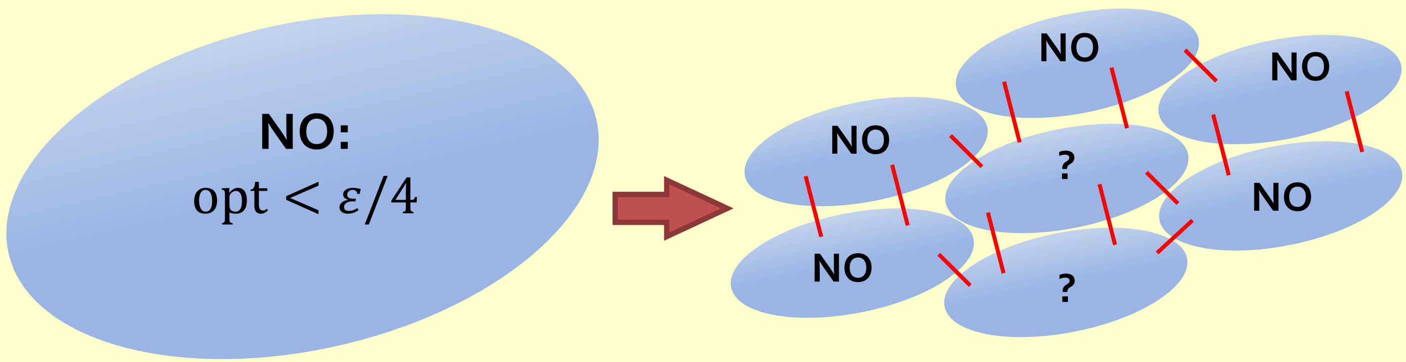

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Easy graphs for UNIQUE GAMES

Expanding constraint graph [Arora-Khot-Kolla-S.-Tulsiani-Vishnoi'08]

UG(ε) in time poly(n) if eigenvalue gap $> 100\varepsilon$

Constraint graph with few large eigenvalues [Kolla-Tulsiani'07, Kolla'10, *here*, Barak-Raghavendra-S.'11]

UG(ε) in time $\exp(m)$ if at most m eigenvalues $> 1 - 100\varepsilon$

quasi-expander

Graph Decomposition

***Idea*: quasi-expander**
≈ small-set expander

Here
~~Classical~~ [Leighton-Rao … nan-Teng'04, Trevisan'05]

By removing 1% of edges, every graph can be decomposed into components with ~~eigenvalue gap $1/\text{polylog}(n)$~~

at most $n^{O(\varepsilon^{1/3})}$ eigenvalues $> 1 - 100\varepsilon$

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$\mathrm{UG}(\varepsilon)$ in time $\exp\left(n^{\varepsilon^{1/3}}\right)$

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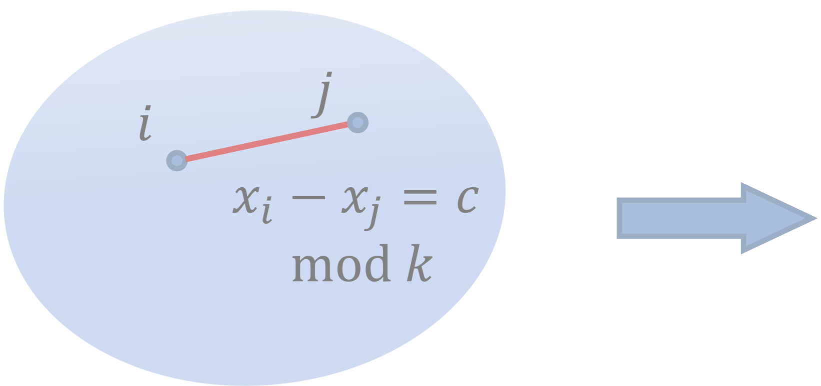
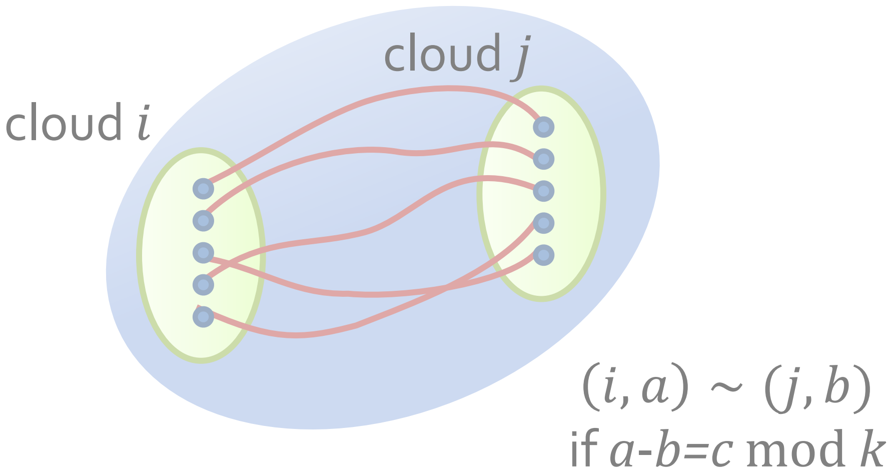
$\mathrm{UG}(\varepsilon)$ in time $\exp(m)$ if at most m eigenvalues $> 1 - 100\varepsilon$

Assume: label-extended graph has at most m eigenvalues $> 1 - 100\varepsilon$

constraint graph G



label-extended graph $\hat{G}$



assignment satisfying $1 - \varepsilon$ of constraints → vertex set of size n and expansion $\leq \varepsilon$

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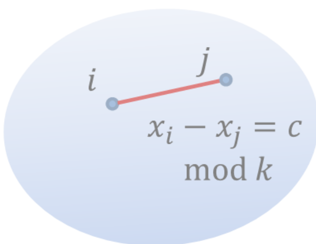

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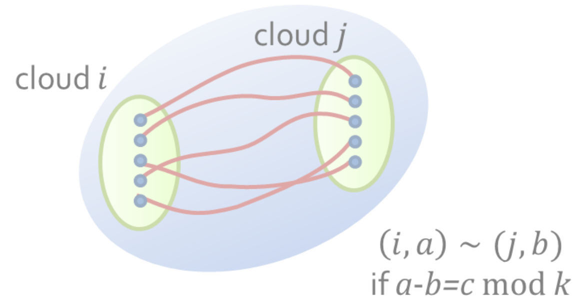


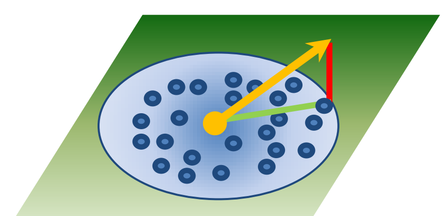
assignment satisfying $1 - \varepsilon$ of constraints → vertex set of size n and expansion $\leq \varepsilon$

↓

99% of indicator vector lies in **span of top m eigenvectors**

enumerate subspace ←

assignment satisfying 90% of constraints



Constraint graph with few large eigenvalues [Kolla-Tulsiani'07, Kolla'10, *here*, Barak-Raghavendra-S.'11]

$\mathrm{UG}(\varepsilon)$ in time $\exp(m)$ if at most m eigenvalues $> 1 - 100\varepsilon$

Assume: label-extended graph has at most m eigenvalues $> 1 - 100\varepsilon$

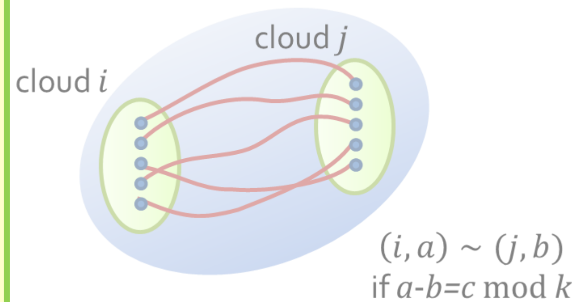
x = normalized indicator vector

Suppose: > 1% of x is orthogonal to **span**

$$\langle x, \hat{G}x \rangle < 0.99\,\lambda_1 + 0.01\,\lambda_{m+1} \leq 1 - \varepsilon$$

$\Longrightarrow$ expansion $> \varepsilon$

label-extended graph $\hat{G}$

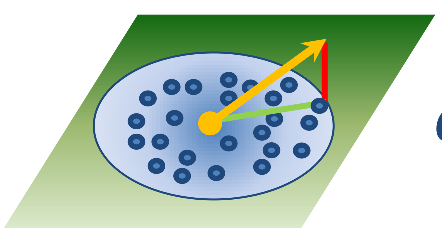


vertex set of size n and expansion $\leq \varepsilon$

99% of indicator vector lies in **span of top m eigenvectors**

enumerate subspace

assignment satisfying 90% of constraints



Compare: Fourier-based learning

Constraint graph with few large eigenvalues [Kolla-Tulsiani'07, Kolla'10, *here*, Barak-Raghavendra-S.'11]

$\mathrm{UG}(\varepsilon)$ in time $\exp(m)$ if at most m eigenvalues $> 1 - 100\varepsilon$

Subexponential Algorithm for Unique Games

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Graph Decomposition

Here

~~Classical~~

Idea: quasi-expander
≈ small-set expander

By removing 1% of edges, every graph can be decomposed into components with ~~eigenvalue gap $1/\mathrm{polylog}(n)$~~

at most $n^{O(\varepsilon^{1/3})}$ eigenvalues $> 1 - \varepsilon$

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Graph Decomposition

***Idea*: quasi-expander $\approx$ small-set expander**

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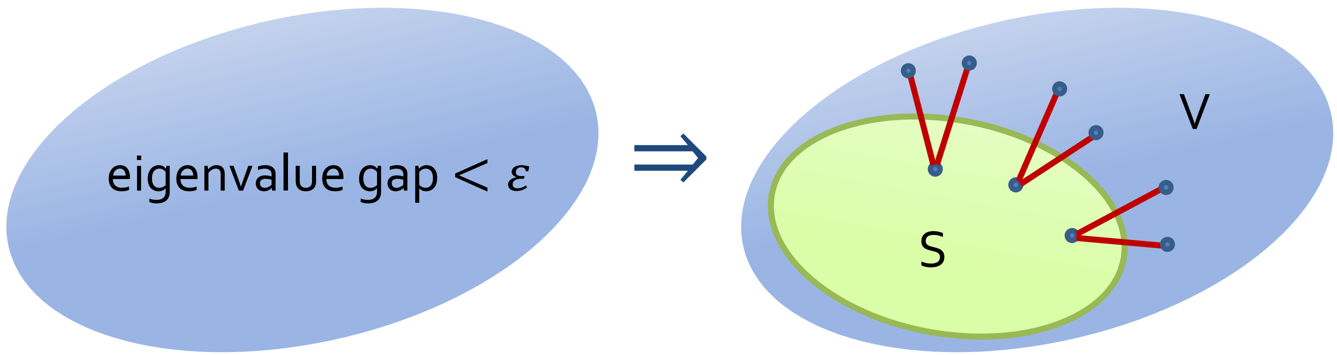
Graph Decomposition

By removing 1% of edges, every graph can be decomposed into components with at most $n^{O(\varepsilon^{1/3})}$ eigenvalues $> 1 - \varepsilon$

Assume graph is d-regular

For $\varepsilon = 1/\text{polylog}\, n$,

follows from Cheeger bound [Cheeger'70, Alon-Milman'85, Alon'86]



non-expanding set S

expansion$(S) < \sqrt{\varepsilon}$

$|S| < |V|/2$

Graph Decomposition

By removing 1% of edges, every graph can be decomposed into components with at most $n^{O(\varepsilon^{1/3})}$ eigenvalues $> 1-\varepsilon$

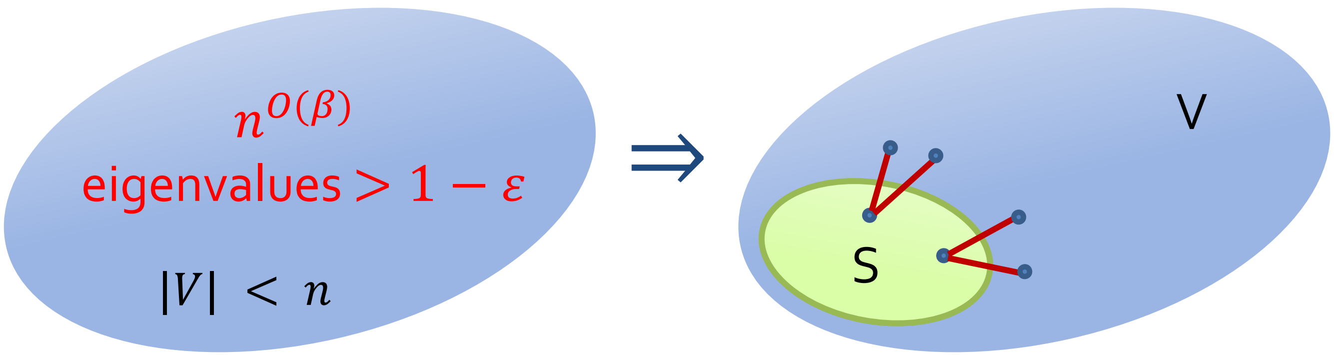
Assume graph is d-regular

general $\varepsilon > 0$

For ~~$\varepsilon = 1/\text{polylog}\, n$~~,

"higher-order Cheeger bound" [here]

follows from ~~Cheeger bound~~



small non-expanding set S

expansion(S) $< \sqrt{\varepsilon/\beta}$

$|S| < |V|/\boldsymbol{n^{\beta}}$

“higher-order Cheeger bound”

$n^{O(\beta)}$

eigenvalues $> 1 - \varepsilon$ $\Rightarrow$

$|V| < n$

expansion(S) $< \sqrt{\varepsilon/\beta}$

$|S| < |V|/n^{\beta}$

"higher-order Cheeger bound"

$n^{O(\beta)}$ eigenvalues $> 1 - \varepsilon$ $\Rightarrow$ expansion$(S) < \sqrt{\varepsilon/\beta}$

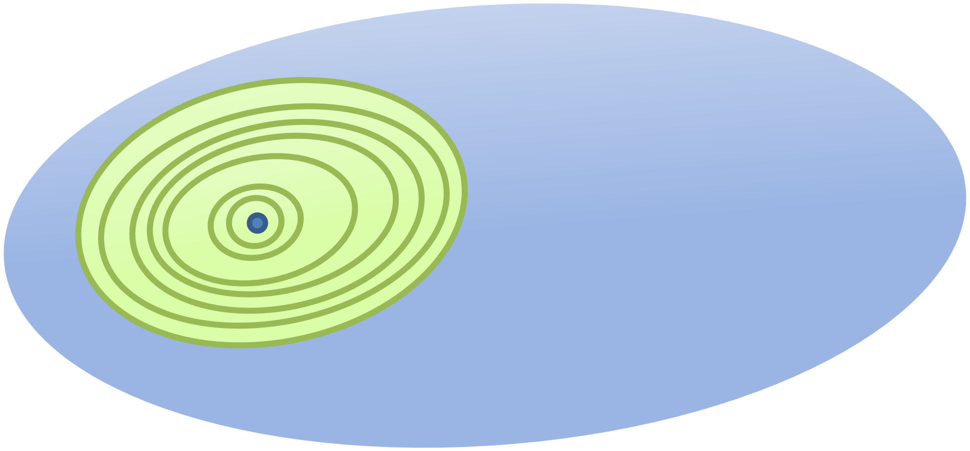
$|V| < n$ $|S| < |V|/n^{\beta}$

"higher-order Cheeger bound"

$n^{O(\beta)}$ eigenvalues $> 1 - \varepsilon$ $\Longrightarrow$ expansion$(S) < \sqrt{\varepsilon/\beta}$

$|V| < n$ $\quad$ $|S| < |V|/n^{\beta}$

Idea



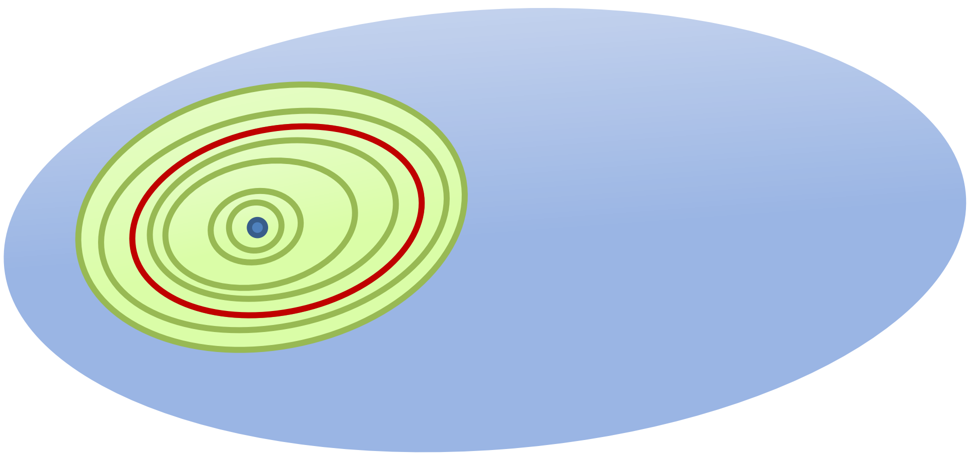
“higher-order Cheeger bound”

$n^{O(\beta)}$ eigenvalues $> 1-\varepsilon$ $\Rightarrow$ expansion$(S) < \sqrt{\varepsilon/\beta}$

$|V| < n$ $\quad$ $|S| < |V|/n^{\beta}$

It would suffice to show: (for degree $d = O(1)$)

$\exists$ vertex i such that $|\mathrm{Ball}(i,t)| < |V|/n^{\beta}$ for $t = (\beta/\varepsilon)\log n$



$\Rightarrow$ volume growth $< 1 + (\varepsilon/\beta)$ in intermediate step $s < t$

$\Rightarrow$ $\mathrm{Ball}(i,s)$ has expansion $< O(\varepsilon/\beta)$

“higher-order Cheeger bound”

$n^{O(\beta)}$ eigenvalues $> 1 - \varepsilon$ $\Longrightarrow$ expansion(S) $< \sqrt{\varepsilon/\beta}$

$|V| < n$ $\quad$ $|S| < |V|/n^{\beta}$

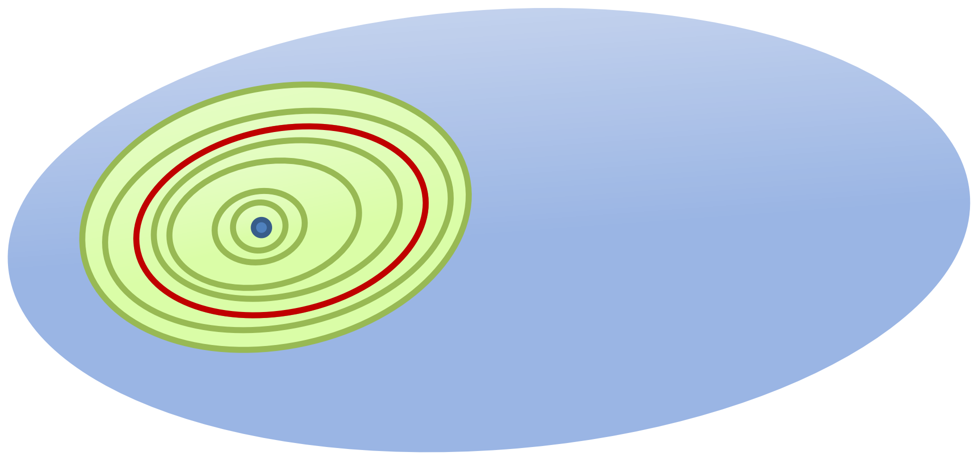
Suffices

~~*It would suffice*~~ *to show*:

$$\|G^t e_i\|^2 > n^{\beta}/|V|$$

collision probability of t-step random walk from i

$\exists$ vertex i such that ~~$|\mathrm{Ball}(i,t)| < |V|/n^{\beta}$~~ for $t = \left({}^{\beta}/{}_{\varepsilon}\right) \log n$



Heuristic:

collision probability $\approx \dfrac{1}{|\text{truncated support}|}$

local Cheeger bound

$\Rightarrow$ *collision probability decay* $< 1 + \left({}^{\varepsilon}/{}_{\beta}\right)$ in intermediate step $s < t$

$\Rightarrow$ level set of $G^s e_i$ has expansion $< \sqrt{\varepsilon/\beta}$ and size $< |V|/n^{\beta}$

"higher-order Cheeger bound"

$n^{O(\beta)}$ eigenvalues $> 1-\varepsilon$ $\Rightarrow$ expansion$(S) < \sqrt{\varepsilon/\beta}$

$|V| < n$ $|S| < |V|/n^{\beta}$

Suffices

~~*It would suffice*~~ *to show*:

collision probability of t-step random walk from i

$$\|G^t e_i\|^2 > n^{\beta}/|V|$$

$\exists$ vertex i such that ~~$|\mathrm{Ball}(i,t)| < |V|/n^{\beta}$~~ for $t = (\beta/\varepsilon)\log n$

Heuristic:

"*A Markov chain with many large eigenvalues cannot mix locally everywhere*"

$$\textstyle\sum_i \|G^t e_i\|^2 = \sum_i \langle e_i, G^{2t} e_i\rangle = \mathrm{Trace}(G^{2t}) > n^{O(\beta)}(1-\varepsilon)^{2t} > n^{\beta}$$

■

$\Rightarrow$ *collision probability decay* $< 1 + (\beta/\varepsilon)$ in intermediate step $s < t$

$\Rightarrow$ level set of $G^s e_i$ has expansion $< \sqrt{\varepsilon/\beta}$ and size $< |V|/n^{\beta}$

"higher-order Cheeger bound"

$n^{O(\beta)}$ eigenvalues $> 1-\varepsilon$ $\quad\Longrightarrow\quad$ expansion$(S) < \sqrt{\varepsilon/\beta}$

$|V| < n$ $\qquad\qquad$ $|S| < |V|/n^{\beta}$

Suffices

~~It would suffice~~ to show:

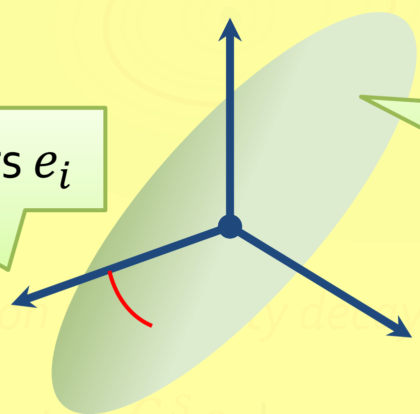
$$\|G^t e_i\|^2 > n^{\beta}/|V|$$

collision probability of t-step random walk from i

$\exists$ vertex i such that ~~$|\mathrm{Ball}(i,t)| < |V|/n^{\beta}$~~ for $t = (\beta/\varepsilon)\log n$

"A Markov chain with many large eigenvalues cannot mix locally everywhere"

vertex indicators e_i



span U of $n^{O(\beta)}$ eigenvectors with eigenvalue $> 1-\varepsilon$

$\Rightarrow \exists$ vertex i.

$$\langle e_i, \mathrm{Proj}_U e_i\rangle \geq \frac{n^{O(\beta)}}{|V|} \qquad \blacksquare$$

More Subexponential Algorithms

Similar approximation for **MULTI CUT** and **SMALL SET EXPANSION**

Better approximations for **MAX CUT** and **VERTEX COVER** on small-set expanders

Improved approximations for **d-TO-1 GAMES** (→ Khot's d-to-1 Conjecture) [S.'10]

Open Questions

What else can be done in subexponential time?

Better approximations for **MAX CUT** or **VERTEX COVER** on general instances?

Example: $f(\varepsilon)$-approximation for **SPARSEST CUT** in time $\exp(n^{\varepsilon})$?

Towards refuting the Unique Games Conjecture

How many large eigenvalues can a small-set expander have?

Is Boolean noise graph the worst case? ($\mathrm{polylog}(n)$ large eigenvalues)

Thank you! Questions?