Semidefinite Programming Hierarchies and the Unique Games Conjecture

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based on joint works with

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Approximability of CSPs, Toronto, August 2011

Semidefinite Programming Hierarchies

[Sherali-Adams'90, Lovász-Schrijver'91, Lasserre'01]

powerful algorithmic technique

Better approximations for combinatorial optimization problems?

Example: **MAX CUT** [Goemans-Williamson'94]

Unique Games Conjecture

[Khot'02,...]

strong implications for *hardness of approximation*

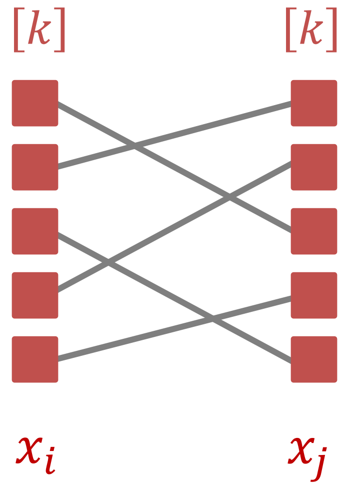
Is the conjecture true or false?

Constraint Satisfaction Problems

[e.g., works on *dense* or *pseudo-dense* instances]

important class of optimization problems

What instances are hard / easy?



UNIQUE GAMES

Input: list of constraints of form $x_i - x_j = c \bmod k$

Goal: satisfy as many constraints as possible

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Unique Games Conjecture (UGC) [Khot'02]

For every $\varepsilon > 0$, the following is NP-hard:

$\mathrm{UG}(\varepsilon)$

Input: **UNIQUE GAMES** instance with $k = \log n$ (say)

Goal: Distinguish two cases

YES: more than $1 - \varepsilon$ of constraints satisfiable

NO: less than ε of constraints satisfiable

Implications of UGC

For many basic optimization problems,
it is NP-hard to beat current algorithms
(based on simple LP or SDP relaxations)

Examples:

VERTEX COVER [Khot-Regev'03],
MAX CUT [Khot-Kindler-Mossel-O'Donnell'04,
Mossel-O'Donnell-Oleszkiewicz'05],
every **MAX CSP** [Raghavendra'08], ...

Implications of UGC

For many basic optimization problems,
it is NP-hard to beat current algorithms
(based on simple LP or SDP relaxations)

Unique Games Barrier

Example: $(\alpha_{\mathrm{GW}}+\varepsilon)$-approximation for **MAX CUT**
at least as hard as $\mathrm{UG}(\varepsilon')$

$\alpha_{\mathrm{GW}} = 0.878\ldots$
Goemans–Williamson
bound for **MAX CUT**

UNIQUE GAMES is common barrier for improving current algorithms of many basic problems

Subexponential Algorithm for Unique Games

$\mathrm{UG}(\varepsilon)$ in time $\exp\left(n^{\varepsilon^{1/3}}\right)$

Time vs Approximation Trade-off

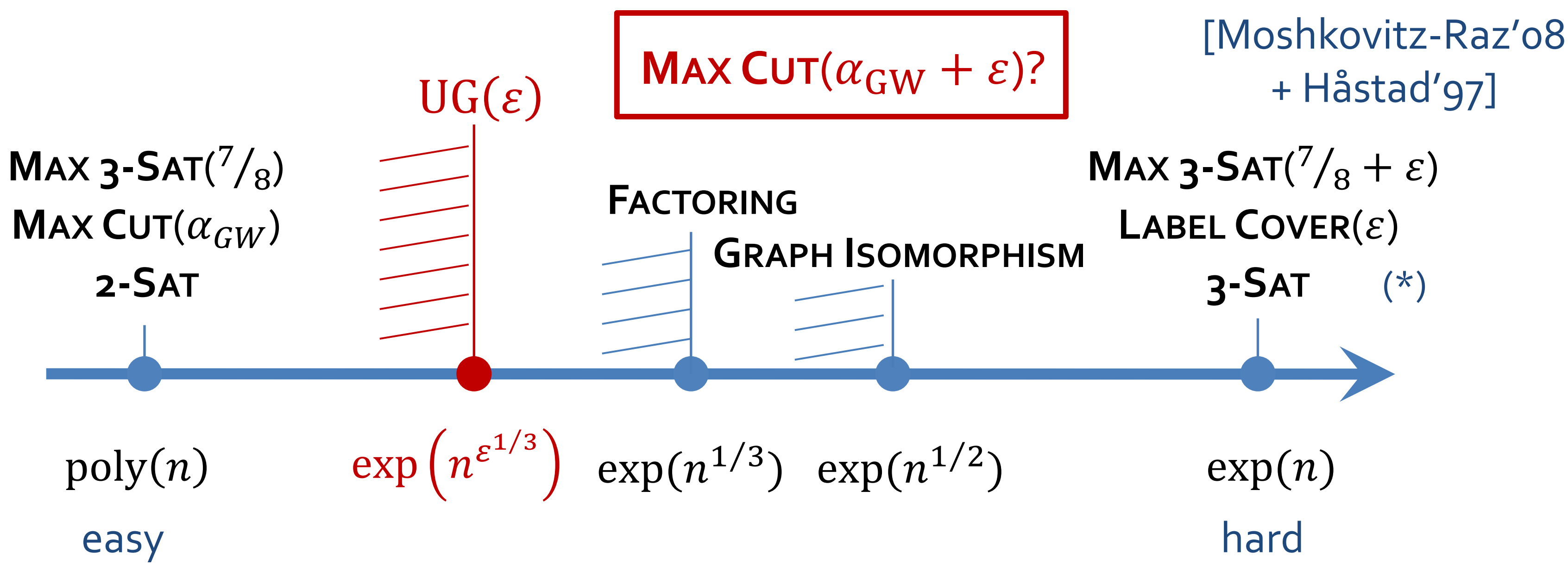
Input: **UNIQUE GAMES** instance with alphabet size k such that $1-\varepsilon$ of constraints are satisfiable,

Output: assignment satisfying $1-C\sqrt{\varepsilon}$ of constraints

Time: $\exp\left(k\, n^{1/C^{2/3}}\right)$

Subexponential Algorithm for Unique Games

UG(ε) in time $\exp\left(n^{\varepsilon^{1/3}}\right)$



(*) assuming *Exponential Time Hypothesis* [Impagliazzo-Paturi-Zane'01]
(3-Sat has no $2^{o(n)}$ algorithm)

Subexponential Algorithm for Unique Games

$\mathrm{UG}(\varepsilon)$ in time $\exp\left(n^{\varepsilon^{1/3}}\right)$

Consequences

NP-hardness reduction for $\mathrm{UG}(\varepsilon)$ must have blow-up $n^{1/\varepsilon^{1/3}}$ (*)
→ rules out certain classes of reductions for proving UGC

Analog of UGC with subconstant ε (say $\varepsilon = 1/\log\log n$) is false (*)
(*contrast*: subconstant hardness for **LABEL COVER** [Moshkovitz-Raz'08])

(*) assuming 3-SAT does not have subexponential algorithms, $\exp(n^{o(1)})$

Subexponential Algorithm for Unique Games

$\mathrm{UG}(\varepsilon)$ in time $\exp\left(n^{\varepsilon^{1/3}}\right)$

here: via semidefinite programming hierarchies [Sherali-Adams'90, Lovász-Schrijver'91, Lasserre'01]

general framework for rounding SDP hierarchies

(especially for "*pseudo-random*" instances of 2-CSPs)

constraint graph has few significant eigenvalues

Unique Games

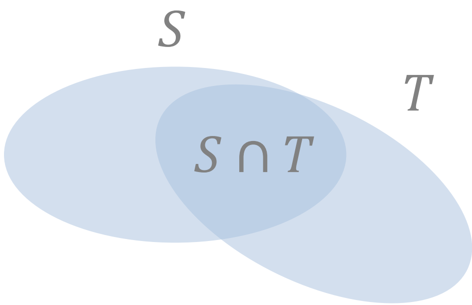
Input: list of constraints of form $x_i - x_j = c \bmod k$

Goal: satisfy as many constraints as possible

What we want:

$X_1, \dots, X_n$ *jointly distributed* random variables over $[k]$

$\Pr(X_i - X_j \equiv c) \geq 1 - \varepsilon$ for typical constraint $x_i - x_j \equiv c$



time $n^{O(r)}$

distributions X_S over $[k]^S$ for all S with $|S| \le r$

consistency

same marginal for $S \cap T$ in distributions X_S and X_T

positive semidefiniteness

X_{ia} = indicator of $\{X_i = a\}$

$\mathrm{Cov}(X_{ia}, X_{jb} \mid X_S = \alpha)$ p.s.d. for all S with $|S| \le r$-2

r-level SDP hierarchy:

~~*What we want:*~~

$X_1, \dots, X_n$ r-local ~~*jointly distributed*~~ random variables over $[k]$

$\Pr(X_i - X_j \equiv c) \ge 1 - \varepsilon$ for typical constraint $x_i - x_j \equiv c$

Goal: produce *global* random variables $X'_1, \dots, X'_n$

$\{X'_i, X'_j\} \approx \{X_i, X_j\}$ for most constraints $x_i - x_j \equiv c$

Here: iterative procedure for $r = n^{O(\varepsilon^{1/3})}$ [Arora-Barak-S.'10 + Barak-Raghavendra-S.'11]

Components of iterative procedure

Rounding

sample variables independently according to their marginals

Conditioning

pick a vertex j and sample X_j

condition $X_1, \ldots, X_n$ on sample for X_j

Partitioning

find vertex subset S

break dependence between X_S and $X_{V \setminus S}$

general framework for rounding SDP hierarchies

(Pairwise) Correlation

$\mathrm{Corr}(X_i, X_j)$ measures how much the distribution of X_i changes when conditioned on X_j

Examples:

$\mathrm{Corr}(X_i, X_j) = 0 \quad \Leftrightarrow \quad X_i$ and X_j independent

$\mathrm{Corr}(X_i, X_j) = 1 \quad \Leftrightarrow \quad X_i$ fixed after fixing X_j

(Pairwise) Correlation

$\mathrm{Corr}(X_i, X_j)$

statistical distance between $\{X_i, X_j\}$ and $\{X_i\}\{X_j\}$

$$\max_c \sum_a \left|\mathrm{Cov}\left(X_{ia}, X_{j(a+c)}\right)\right|$$

decrease in variance when conditioning on X_j

$$\sum_a \left(Var(X_{ia}) - Var(X_{ia}|X_j)\right)$$

similar to *mutual information*
$I(X;Y) = H(X) - H(X \mid Y)$

Important fact:

can approximate $\mathrm{Corr}(X_i, X_j)$ by Gram matrix of unit vectors (tensoring trick)

Rounding	*Conditioning*	*Partitioning*

sample variables independently according to their marginals

$\mathrm{Corr}(X_i, X_j) \approx$ statistical distance between independent and correlated sampling

If $\mathrm{Corr}(X_i, X_j) < 1 - O(\varepsilon)$ then *independent sampling* satisfies constraint with probability $\Omega(\varepsilon)$

Rounding fails $\Rightarrow$ $\mathbf{E}_{i \sim j}\mathrm{Corr}(X_i, X_j) > 1 - O(\varepsilon)$

Local Correlation
(over edges of constraint graph)

pick a vertex j and sample X_j

condition $X_1, \dots, X_n$ on sample for X_j

Issue:

computationally expensive

(level r $\rightarrow$ level $r-1$)

$\mathrm{Corr}(X_i, X_j)$ measures decrease in variance when conditioning on X_j

Idea:

condition on vertex j only if $\mathbf{E}_i \mathrm{Corr}(X_i, X_j) > n^{-\beta}$

$\Rightarrow$ can condition at most n^{β} times on such vertices

Conditioning fails $\Rightarrow$ $\mathbf{E}_{i,j} \mathrm{Corr}(X_i, X_j) < n^{-\beta}$

Global Correlation
(over random vertex pairs)

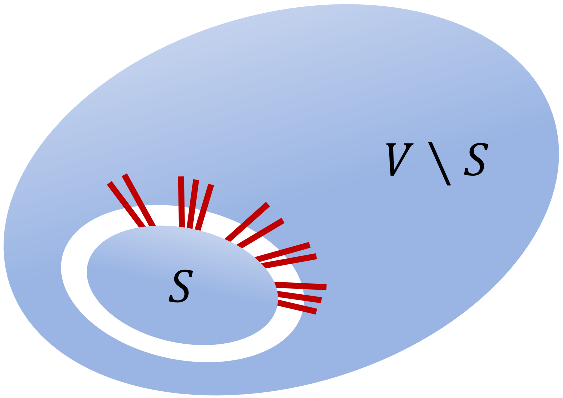
find vertex subset S

break dependence between X_S and $X_{V \setminus S}$

Issue:

destroys correlation for constraints between S and $V \setminus S$

in total: should break at most $^1/_2$ of the constraints



***Wanted**:*

set S with *small expansion*

& *small cardinality*

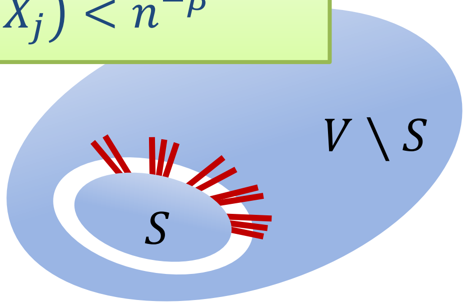
Rounding *Conditioning* ***Partitioning***

fails only if *local correlation high*
$\mathbf{E}_{i\sim j}\mathrm{Corr}(X_i, X_j) > 1 - O(\varepsilon)$

fails only if *global correlation low*
$\mathbf{E}_{i,j}\mathrm{Corr}(X_i, X_j) < n^{-\beta}$

find vertex subset S

break dependence between X_S and $X_{V\setminus S}$



Correlation Propagation

For endpoints of random path of length $t = \Omega(\beta/\varepsilon \cdot \log n)$

$\mathbf{E}_{i\sim_t j}\mathrm{Corr}(X_i, X_j) > (1 - O(\varepsilon))^t \gg n^{-\beta/2}$

$\mathrm{Corr}(X_i, X_j)$ is Gram matrix of unit vectors

$\Rightarrow$ random walk stuck for t steps on $n^{-\beta/2}$ fraction of vertices

$\Rightarrow \exists$ vertex set S with $|S| < |V|/n^{\beta/2}$ and expansion $O\left(\sqrt{\varepsilon/\beta}\right)$

A vertex is cut in $\leq 2/\beta$ partitioning steps $\Rightarrow$ break only $O\left(\sqrt{\varepsilon/\beta^3}\right)$ edges

■

Can we avoid partitioning?

Yes, for a large family of instances!

Local vs Global Correlation and Higher Eigenvalues

$$\mathbf{E}_{i\sim j}\mathrm{Corr}(X_i, X_j) > \lambda$$
$$\mathbf{E}_{i,j}\mathrm{Corr}(X_i, X_j) < 1/r$$

$\Rightarrow$ constraint graph has at least $\varepsilon \cdot r$ eigenvalues larger than $\lambda - \varepsilon$

Rounding — *Conditioning* — *Partitioning*

fails only if *local correlation high*
$\mathbf{E}_{i\sim j}\mathrm{Corr}\big(X_i, X_j\big) > 1 - O(\varepsilon)$

fails only if *global correlation low*
$\mathbf{E}_{i,j}\mathrm{Corr}\big(X_i, X_j\big) < 1/r$

Local vs Global Correlation and Higher Eigenvalues

$\mathbf{E}_{i\sim j}\mathrm{Corr}\big(X_i, X_j\big) > \lambda$

$\mathbf{E}_{i,j}\mathrm{Corr}\big(X_i, X_j\big) < 1/r$

$\Rightarrow$ constraint graph has at least $\varepsilon \cdot r$ eigenvalues larger than $\lambda - \varepsilon$

Corollary

r^{th} levels of hierarchy solves $\mathrm{UG}(\varepsilon)$ if constraint graph has at most $\varepsilon \cdot r$ eigenvalues larger than $1 - O(\varepsilon)$

[improves Kolla'10]

Rounding — *Conditioning* — *Partitioning*

fails only if *local correlation high*
$\mathbf{E}_{i\sim j}\mathrm{Corr}(X_i, X_j) > O(\eta)$

fails only if *global correlation low*
$\mathbf{E}_{i,j}\mathrm{Corr}(X_i, X_j) < 1/r$

Local vs Global Correlation and Higher Eigenvalues

$\mathbf{E}_{i\sim j}\mathrm{Corr}(X_i, X_j) > \lambda$

$\mathbf{E}_{i,j}\mathrm{Corr}(X_i, X_j) < 1/r$

$\Rightarrow$ constraint graph has at least $\varepsilon \cdot r$ eigenvalues larger than $\lambda - \varepsilon$

Approximation Scheme for ***Max Cut***

[improves works on (pseudo-)dense problems: Arora-Karger-Karpinski'95, Fernandez de la Vega'96, ...]

r^{th} levels of hierarchy finds optimal cut up to $\mathrm{O}(\eta)$ fraction of edges if graph has at most $\eta \cdot r$ eigenvalues larger than η

canonical SDP integrality gap instances have rapidly decaying eigenvalues ($r = \mathrm{polylog}(n)$)

More SDP-hierarchy algorithms

For general **2-CSP**: [Barak-Raghavendra-S.'11]

PTAS if constraint graph is random (degree $\gg$ alphabet)

Subsequent works:

better **3-COLORING** approximation on some graph families [Arora-Ge'11]

better approximations for **MAX-BISECTION** [Raghavendra-Tan'11]

Independent work: [Guruswami-Sinop'11]

approximation schemes for quadratic integer programming
with p.s.d. objective & few relevant eigenvalues

Open Questions

What else can be done in subexponential time?

Better approximations for **Max Cut**, **Vertex Cover** on general instances?

Example: $f(\varepsilon)$-approximation for **Sparsest Cut** in time $\exp(n^{\varepsilon})$?

Towards settling the Unique Games Conjecture

SDP integral gap instances with $\mathrm{poly}(n)$ large eigenvalues?

Recent progress: [Barak-Gopalan-Håstad-Meka-Raghavendra-S.'11]

∃ gap instances with $\mathrm{qpoly(n)} = 2^{(\log n)^{\Omega(1)}}$ large eigenvalues

Also: gap remains for $\mathrm{qpoly}(n)$ levels of an SDP hierarchy

Thank you! Questions?