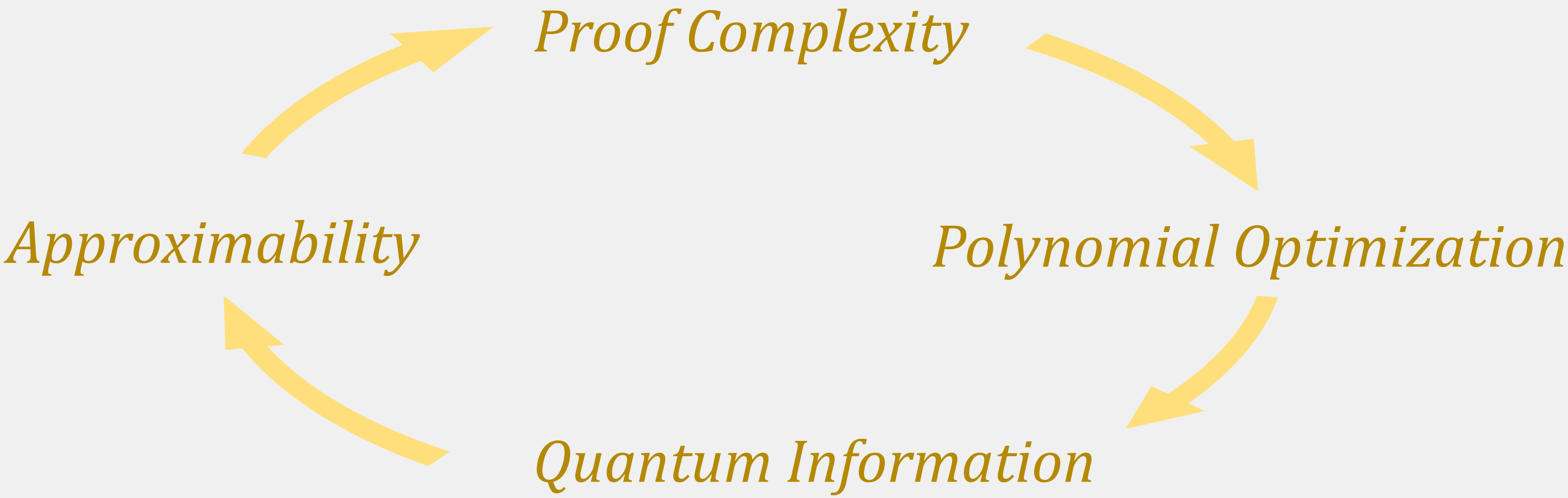
Unique Games Conjecture & Polynomial Optimization

David Steurer

Newton Institute, Cambridge, July 2013

overview


Proof Complexity
Polynomial Optimization
Quantum Information
Approximability

computational complexity & approximation

What approximations can efficient methods guarantee?

(in the worst-case)

many basic optimization problems are NP-hard [Karp'71]

$\Rightarrow$ efficient methods cannot be exact (require* time $2^{n^{\Omega(1)}}$)

But optimization is not all-or-nothing!

approximation reveals rich structure — both for algorithms and hardness

connections to harmonic analysis, metric geometry, error-correcting codes, …

goal: understand *computational* price of *approximation* quality

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* under standard complexity assumptions, $\mathrm{NP} \not\subseteq \mathrm{TIME}\left(2^{n^{o(1)}}\right)$

What approximations can efficient methods guarantee?

example: MAX 3-XOR

given: linear equations modulo 2, each with three variables
find: assignment that satisfies as many as possible

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 1 \\
x_4 + x_5 + x_6 &= 0 \\
&\vdots \\
x_2 + x_4 + x_6 &= 1 \\
x_{99} + x_{33} + x_{77} &= 0
\end{aligned}
$$

What approximations can efficient methods guarantee?

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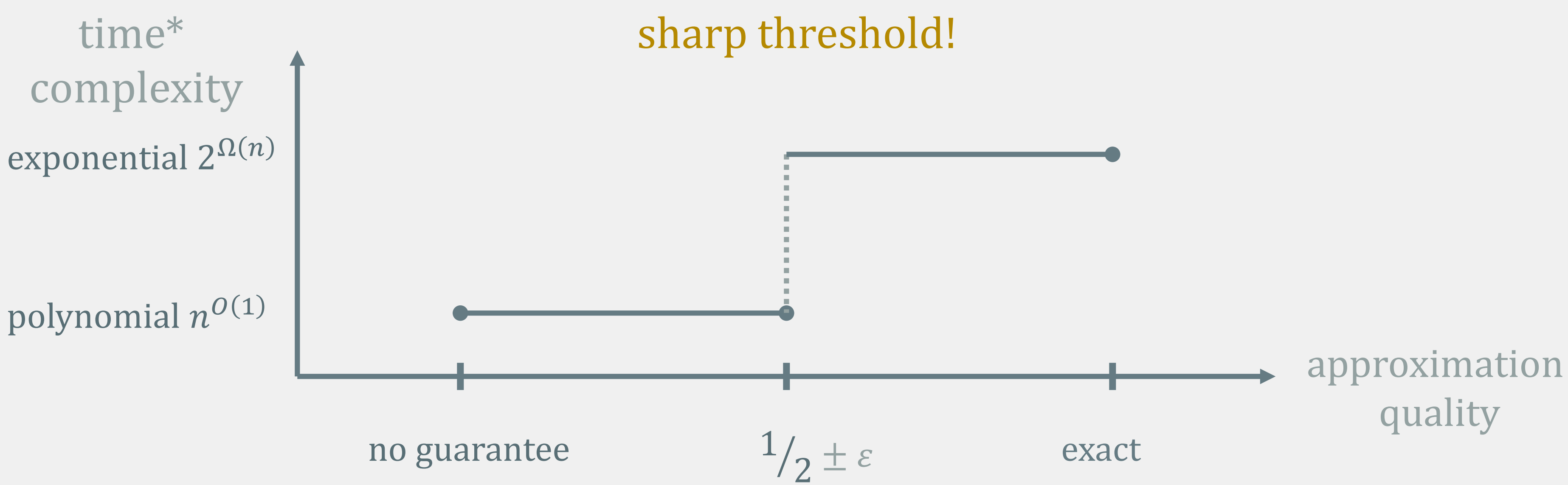


* under standard complexity assumptions, 3SAT $\notin$ TIME$\left(2^{o(n)}\right)$

What approximations can efficient methods guarantee?

example: MAX 3-XOR

[culmination of many works]

many ideas and techniques (error correction, random walks in graphs, harmonic analysis, ...)

[e.g., see Arora-Barak textbook]

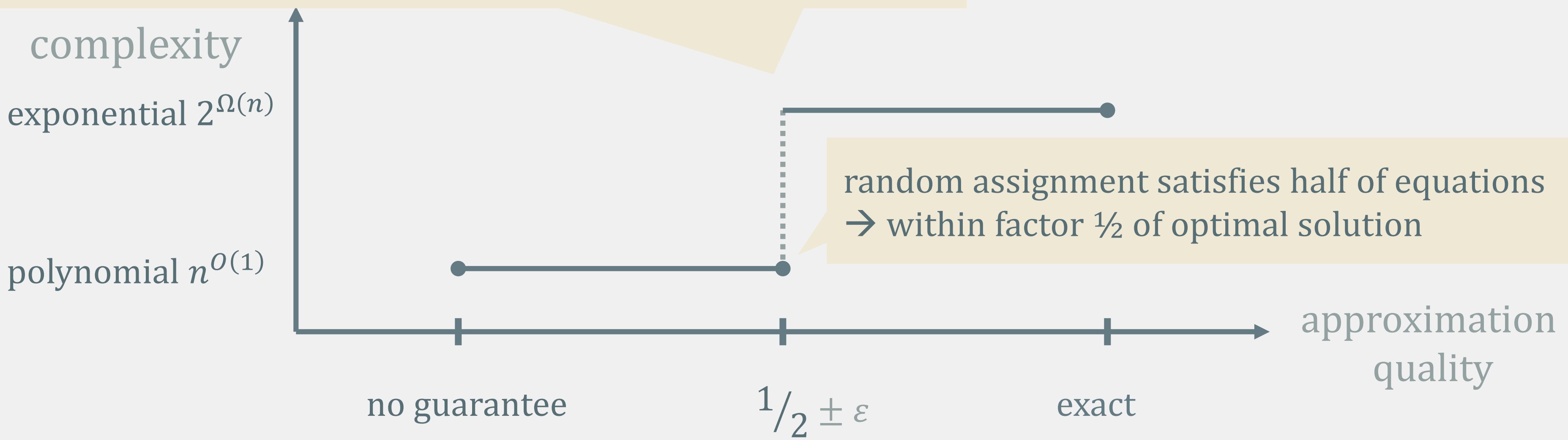
reduction from 3-SAT to MAX 3-XOR: (linear-time!)

satisfiable → $(1-\varepsilon)$-satisfiable

not satisfiable → $(^1/_2+\varepsilon)$-satisfiable

each with three variables

s many as possible



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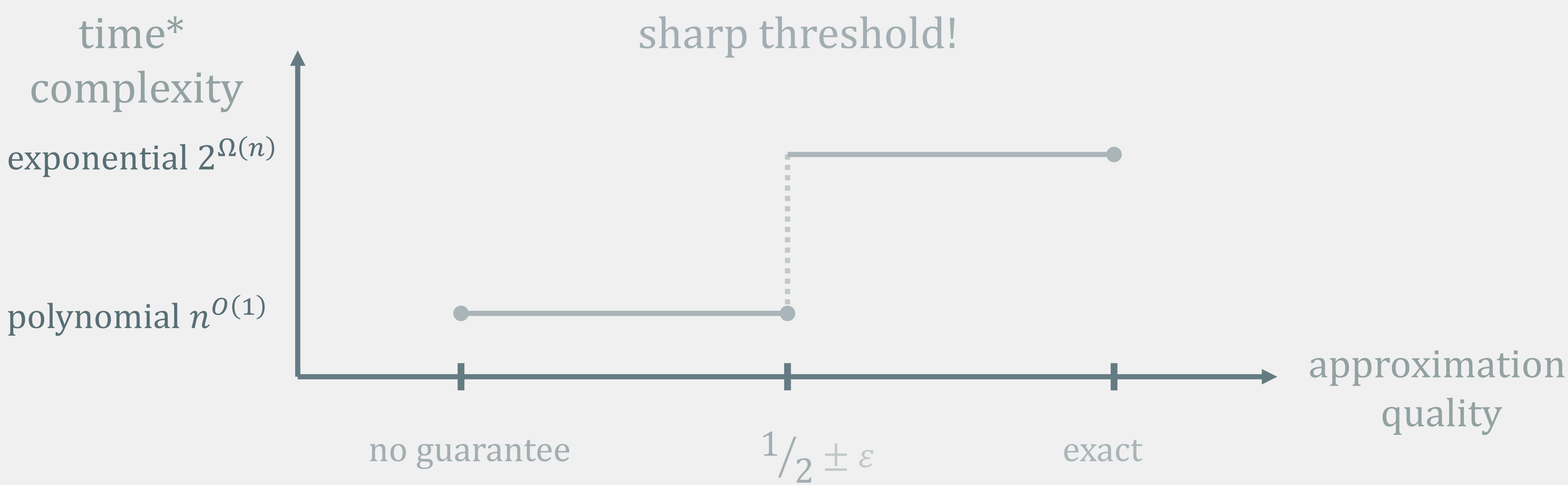
What approximations can efficient methods guarantee?

example: MAX 3-XOR

What about other problems? ***Same picture?***

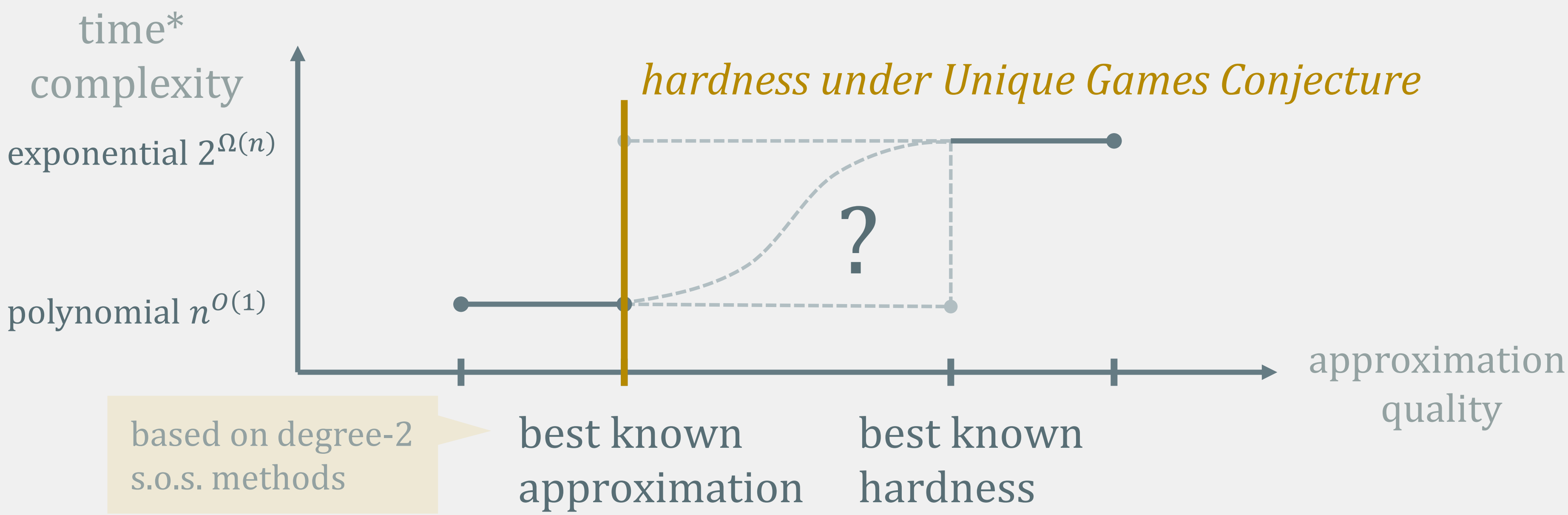
e.g., MAX 2-XOR / MAX CUT

given: linear equations modulo 2, each with three variables
find: assignment that satisfies as many as possible



* under standard complexity assumptions, $3\mathrm{SAT} \notin \mathrm{TIME}\left(2^{o(n)}\right)$

What approximations can efficient methods guarantee?



for most basic optimization problems: (e.g., MAX 2-XOR / MAX CUT)

big gap between known algorithms and known hardness results

What's the trade-off in this window? *What algorithms achieve it?*

under Unique Games Conjecture: current approximations NP-hard to beat!

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* under standard complexity assumptions, $3SAT \notin TIME(2^{o(n)})$

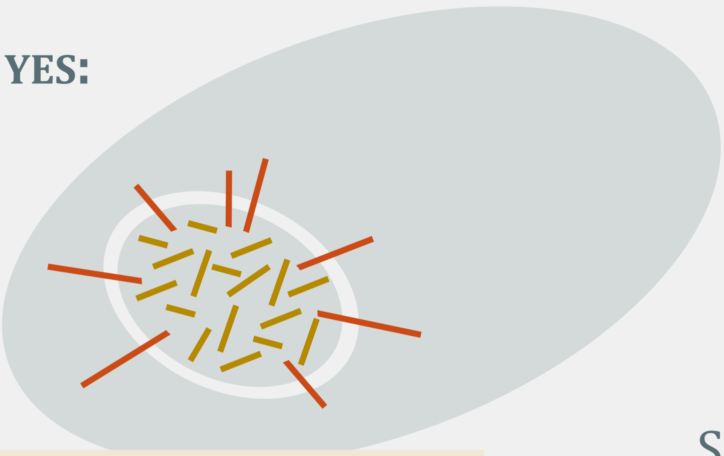
Unique Games Conjecture (UGC) [Khot'02]

Cheeger's bound solves problem for non-small sets

Small-Set Expansion Hypothesis (≈ graph version of UGC):
for every constant $\varepsilon > 0$, it is NP-hard to distinguish two cases:

$\text{SSE}(\varepsilon)$

YES:



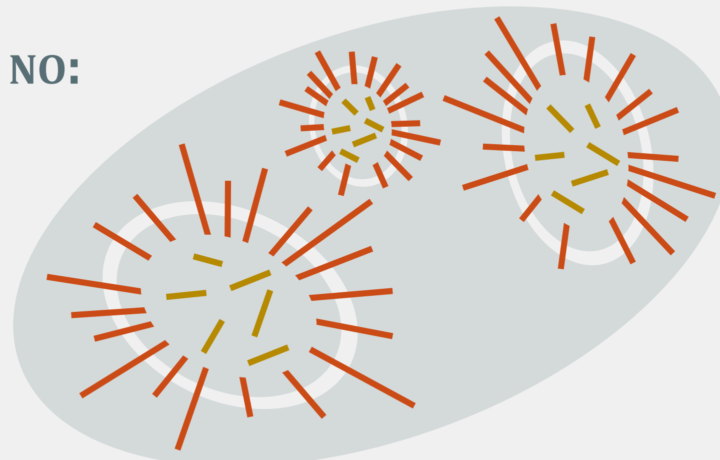
$2^{-O(1/\varepsilon)}$ fraction of vertices

exists small set
with expansion $\leq \varepsilon$

$\frac{|E(S,\bar{S})|}{|E(S,V)|}$

(small cult)

NO:



every small set has
expansion $\geq 1 - \varepsilon$

(locally well-connected)

Unique Games Conjecture (UGC) [Khot'02]

Small-Set Expansion Hypothesis (≈ graph version of UGC): [Raghavendra-S.'10]
for every constant $\varepsilon > 0$, it is NP-hard to solve SSE(ε).

implications

for **large classes of problems**, current approximations NP-hard to beat
[culmination of many works]

UGC →
- constraint satisfaction problems (e.g., MAX CUT)
- strict monotone CSPs (e.g., VERTEX COVER)
- ordering (e.g., MAX ACYCLIC SUBGRAPH, MINIMUM LINEAR ARRANGEMENT)
- metric labeling problems (e.g., MULTIWAY CUT)
- graph partitioning (e.g., BALANCED SEPARATOR, MIN k-CUT)
- Grothendieck-type problems

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implications

for **large classes of problems**, current approximations NP-hard to beat

unconditional consequences

unification of current approximation algorithms [e.g., Raghavendra-S.'09]

UGC identifies **common barrier** for improving current approximations

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techniques

squares of low-degree polynomials
have small variance over the hypercube

hypercontractivity:

$$\left(\mathbb{E}_{\{\pm 1\}^n} f^4\right)^{\frac{1}{4}} \leq 3^d \cdot \left(\mathbb{E}_{\{\pm 1\}^n} f^2\right)^{\frac{1}{2}}$$

for all n-variate degree-d polynomials f

$\forall i.\ \left\| \frac{\partial}{\partial x_i} f \right\|_2 \ll \|f\|_2$

invariance principle:
[Mossel-O'Donnell-Oleszkiewicz'05]

$f = X_1 + \cdots + X_n$
→ central limit theorem

Suppose f has low-degree and no influential variable.
Then, for i.i.d. random variables $X_1, \dots, X_n$,
distribution of $\{f(X_1, \dots, X_n)\}$ depends only
on first two moments of X_i.

f cannot distinguish hypercube and sphere

Unique Games Conjecture (UGC) [Khot'02]

Small-Set Expansion Hypothesis (≈ graph version of UGC): [Raghavendra-S.'10]
for every constant $\varepsilon > 0$, it is NP-hard to solve SSE(ε).

true or false?

most known hardness results rule* out these kind of algorithms

no algorithm known to refute it

subexponential-time algorithms solve SSE(ε) in time $2^{n^{O(\varepsilon)}}$
[Arora-Barak-S.'10] (based on s.o.s.)

UGC predicts beautifully simple complexity landscape

candidate algorithm works for all proposed hard instances
[Barak-Brandao-Harrow-Kelner-S.-Zhou'12] (based on s.o.s.)

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* under standard complexity assumptions, NP $\not\subseteq$ TIME$\left(2^{n^{o(1)}}\right)$

connection to polynomial optimization

best known approximations based on degree-2 sum-of-squares methods

example: MAX CUT $\max_{\{\pm 1\}^n} \frac{1}{4} \sum_{i \sim j} (X_i - X_j)^2$

best-known approximation

find smallest c such that $c - \frac{1}{4} \sum_{i \sim j} (X_i - X_j)^2 = \sum_r R_t^2 + \sum_i \alpha_i (X_i^2 - 1)$

$c - \frac{1}{4} \sum_{i \sim j} (X_i - X_j)^2$ is s.o.s. modulo span$\{X_i^2 - 1 \mid i \in [n]\}$

semidefinite program of size n^2 → polynomial-time algorithm

(actually $\tilde{O}(n)$-time)

[Arora-Kale'07,S.'10]

best c is always within $\alpha_{\mathrm{GW}} \approx 0.878$ factor of optimum value

[Goemans-Williamson]

Can do the same for larger degree! What's the approximation?

connection to polynomial optimization

best known approximations based on degree-2 sum-of-squares methods

example: MAX CUT

$$\max_{\{\pm 1\}^n} \frac{1}{4} \sum_{i\sim j} (X_i - X_j)^2$$

better?

~~*best known*~~ *approximation*

n^d-dimensional linear subspace

find smallest c such that

degree-d part of Ideal($\{\pm 1\}^n$)

$c - \frac{1}{4} \sum_{i\sim j} (X_i - X_j)^2$ is s.o.s. modulo ~~$\mathrm{span}\{X_i^2 - 1 \mid i \in [n]\}$~~

semidefinite program of size n^d → $n^{O(d)}$-time algorithm

[Parrilo, Lasserre '00]

exact for $d = O(n)$:

Suppose $P \geq 0$ over the hypercube.

Interpolate $\sqrt{P}$ over the hypercube → degree-n polynomial Q.

Then, $P = Q^2$ modulo Ideal($\{\pm 1\}^n$). □

connection to polynomial optimization

best known approximations based on degree-2 sum-of-squares methods

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semidefinite program of size n^d → $n^{O(d)}$-time algorithm

[Parrilo, Lasserre '00]

UGC predicts strong limitation of sum-of-squares

for many instances, **degree-$n^{o(1)}$** s.o.s. as bad as degree-2!

But: we don't know instances with **degree-4** as bad as degree-2!

[Barak-Raghavendra-S.'11, Guruswami-Sinop'11]

hard instances?

$\leq k$ extreme eigenvalues → degree-k s.o.s. works
known upper bound: $\leq n^{O(\varepsilon)}$ eigenvalues $\geq 1-\varepsilon$

[Arora-Barak-S.'10]

minimum requirements on underlying graph:

small-set expander (locally well-connected)
many extreme eigenvalues, close to 1 (preferably $n^{\Omega(1)}$ eigenvalues)

fool many weaker relaxation hierarchies

only few constructions known:

[Raghavendra-S.'09, Khot-Saket'09]

Cayley graphs over $\mathbb{F}_2^n$ [Barak-Gopalan-Hastad-Meka-Raghavendra-S.'10]
edge set based on special *error-correcting codes* (locally testable)
analysis based on ***hypercontractivity***

turns out: sum-of-squares solve these instances with **degree ≤ 16**

[Barak-Brandao-Harrow-Kelner-S.-Zhou'12]

connection to proof complexity

sum-of-squares proof system [Grigoriev-Vorobjov'99]

general idea: starting from set of axioms, derive inequalities by applying simple rules

$$P_1 \geq 0, P_2 \geq 0, \ldots, P_m \geq 0$$

proof system is complete*
(*Positivstellensatz*)

$$Q \geq 0$$

derivation rules:

$$\frac{P \geq 0, R \geq 0}{P + R \geq 0} \qquad \frac{P \geq 0, Q \geq 0}{P \cdot Q \geq 0}$$

$$\frac{\emptyset}{R^2 \geq 0}$$

degree of s.o.s. proof ≔ maximum degree** of intermediate polynomial

minimum degree of s.o.s. proof ≃ degree required by s.o.s. method

* for refutations, i.e., $Q = -1$

** for slightly non-standard notion of degree

connection to proof complexity

sum-of-squares proof system [Griogiev-Vorobjov’99]

$$P_1 \geq 0, P_2 \geq 0, \dots, P_m \geq 0$$

$$Q \geq 0$$

derivation rules:

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$$\frac{\emptyset}{R^2 \geq 0}$$

low-degree s.o.s. proofs appear to be powerful and intuitive

[Barak-Brandao-Harrow -Kelner-S.-Zhou’12]

prove good bounds on optimal value
of proposed hard instances with degree ≤ 16
(in particular, *hypercontractivity* and *invariance principle*)

original proofs were *almost* low-degree s.o.s.

connection to quantum information and functional analysis

analytical characterization of small-set expansion

[Barak-Brandao-Harrow-Kelner-S.-Zhou'12]

all sets of volume δ have expansion $1-\lambda^{\Theta(1)}$

$\Leftrightarrow$ 2-to-4 norm of P_λ is bounded by $1/\delta^{\Theta(1)}$

$\max_{f\neq 0}\frac{\|Pf\|_4}{\|f\|_2}$

projector into span of eigenfunctions with eigenvalue $\geq \lambda$

How hard is it to approximate the 2-to-4 norm of a projector?

connection to quantum information and functional analysis

analytical characterization of small-set expansion by 2-to-4 norm $\max_{f \neq 0} \frac{\|Pf\|_4}{\|f\|_2}$

How hard is it to approximate the 2-to-4 norm of a projector?

we don't know … somewhere between time $n^{\Omega(\log n)}$ and $2^{O(\sqrt{n})}$

[Barak-Brandao-Harrow -Kelner-S.-Zhou'12]

close connection to ***quantum separability***

$$\max_{\|u\|=\|v\|=1} \langle u \otimes v, M(u \otimes v) \rangle$$

for operator M with $\|M\|_{2\to 2} \leq 1$

degree-$O(\log n)$ for many special cases

[Brandao-Christiandl-Yard'11, Brandao-Harrow'13, Barak-Kelner-S.'13]

to be continued …

open questions

obvious open questions:

Does s.o.s. give better approximations?
Does s.o.s. refute the Unique Games Conjecture?
Is the Unique Games Conjecture true?

[Grigoriev'00, Schoenebeck'08]

less-obvious open question:

known existence proofs are conjectured to be inherently non-constructive

construct explicit instances with significant approximation gap for low-degree s.o.s. methods

informally: *What short proofs are not even approximately captured by low-degree sum-of-squares proofs?*

Thank you! Questions?