Hypercontractivity, Sum-of-Squares Proofs, and their Applications

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Motivation

Unique Games Conjecture (UGC) [Khot'02]

For every $\varepsilon > 0$, the following is **NP**-hard:

UG(ε)

Given: system of equations $x_i - x_j = c \bmod k$ (say $\mathrm{k} = \log n$)

Distinguish:

YES: at least $1 - \varepsilon$ of equations satisfiable

NO: at most ε of equations satisfiable

Motivation

Unique Games Conjecture (UGC)   [Khot'02]

Implications of UGC

For large class of problems, BASIC SDP achieves optimal approximation

Examples: MAX CUT, VERTEX COVER, any MAX CSP

[Khot-Regev'03, Khot-Kindler-Mossel-O'Donnell'04,
Mossel-O'Donnell-Oleszkiewicz'05, Raghavendra'08]

Is the conjecture true?

In this work:

1) *Evidence:* ∃ polynomial-time algorithm refuting UGC

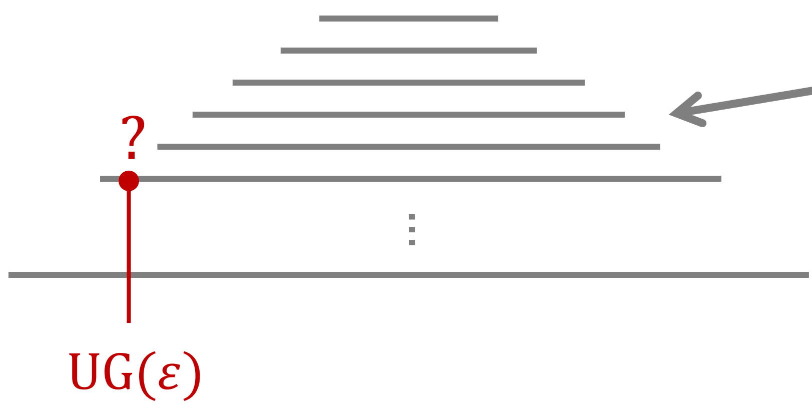
Show: natural algorithm solves all known UG instances (including hard instances for other algorithms)

2) *Evidence:* ∄ polynomial-time algorithm refuting UGC

Show: natural generalization of UG requires qpoly(n)-time (but still admits "same" subexponential algorithm as UG)

Semidefinite Programming (SDP) Hierarchies

[Sherali-Adams'90, Lovász-Schrijver'91,...]



level-k SDP relaxation, time $n^{O(k)}$

Known bounds (for *certain* SDP hierarchies)

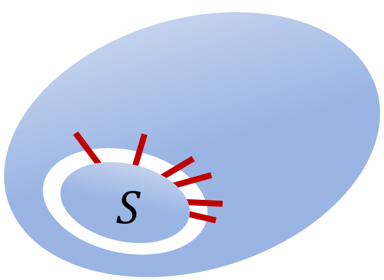
level of $\mathrm{UG}(\varepsilon)$

- $\leq n^{O(\varepsilon^{1/3})}$ [Arora-Barak-S.'10, Barak-Raghavendra-S.'11]
- $\geq \mathrm{qpoly}(n)$ [Raghavendra-S.'09, Khot-Popat-Saket'10, Barak-Gopalan-Håstad-Meka-Raghavendra-S.'11]

$2^{(\log n)^{\Omega(1)}}$

How many significant eigenvalues can a small-set expander have?

(related to Locally Testable Codes)



Semidefinite Programming (SDP) Hierarchies

[Sherali-Adams'90, Lovász-Schrijver'91,...]

level-k SDP relaxation, time $n^{O(k)}$

Result in this work:

aka: Lasserre hierarchy

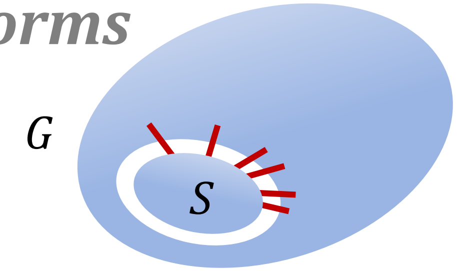
Sum-of-Squares (SoS) hierarchy [Parrilo'00, Lasserre'01]

All *known* UG instances in level-8 of this hierarchy

qualitative difference: basis independence of SoS hierarchy

Small-Set Expansion (SSE) & Operator Norms

(closely related to UG [Raghavendra-S.'09])



Result:

$P_\lambda =$ projector into span of eigenfunctions of G with eigenvalue $\geq \lambda$

all sets of volume $\leq \delta$ have expansion $\geq 1 - \lambda^{\Theta(1)}$

$\Leftrightarrow$ 2-to-4 operator norm of P_λ is $\leq 1/\delta^{\Theta(1)}$ (*hypercontractive*)

$\|P\|_{2\to4} = \max_{f:V\to\mathbb{R}} \|Pf\|_4/\|f\|_2$

f is δ-sparse "$\Leftrightarrow$" $\|f\|_4/\|f\|_2 > 1/\delta^{1/4}$

Corollary: SSE-hard to certify hypercontractivity (even for projectors)

Complexity of Hypercontractivity

Given: projector P into subspace of functions $f: V \to \mathbb{R}$ with $|V| = n$

Promise: $\|P\|_{2\to 4} = O(1)$ (*hypercontractive*)

Certify: $\|P\|_{2\to 4} = O(1)$ (for different constant $O(1)$)

Results:

subexponential time $\exp(n^{1/2})$ suffices (can recover best algorithm for SSE by choice of norm)

quasipolynomial time necessary (*) (builds on hardness of *quantum separability*) [Harrow-Montanaro'10]

(*) assuming **3 SAT** requires $2^{\Omega(n)}$ time

PROOF IDEAS

Result:

(SDP completeness
& integral soundness)

Level-8 SoS relaxation refutes UG instances
based on *long-code* and *short-code* graphs

How to prove it? (rounding algorithm?)

Interpret dual as proof system

Lift soundness proofs to this proof system

Sum-of-Squares Proof System (informal)

Axioms

$P_1(z) \geq 0$

$\vdots$

$P_m(z) \geq 0$

derive

$Q(z) \leq c$

($P_1, \dots, P_m, Q$ bounded-degree polynomials)

Rules

Polynomial operations

$R(z)^2 \geq 0$ for any polynomial R

} "Positivstellensatz" [Stengel'74]

Intermediate polynomials have *bounded degree*

(c.f. bounded-width resolution, but basis independent)

Example

In SoS proof system, $\{z^2 \leq z\} \Leftrightarrow \{0 \leq z \leq 1\}$

Axiom: $z^2 \leq z$ Derive: $z \leq 1$

$$1 - z = z - z^2 + (1 - z)^2$$

$$\geq z - z^2 \qquad \text{(non-negativity of squares)}$$

$$\geq 0 \qquad \text{(axiom)}$$

Components of soundness proof (for known UG instances)

Non-serious issues:

Cauchy–Schwarz / Hölder

Influence decoding

Independent rounding

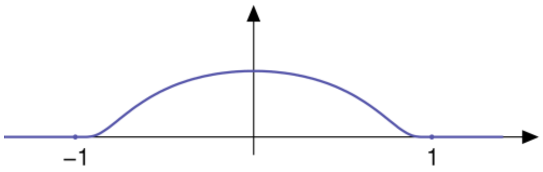
Serious issues:

Hypercontractivity

can use variant of inductive proof, works in *Fourier basis*

Invariance Principle

typically uses *bump functions*, but for UG, polynomials suffice



Concrete component:

Level-4 SoS relaxation certifies
small-set expansion of *long-code* graph

long-code graph

$\mathrm{G} = \mathrm{Cay}(\mathbb{F}_2^m, T)$ where T = {points with Hamming weight εm}

Small-Set Expansion (SSE)

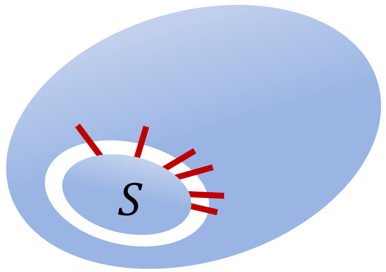
Given: regular graph G with vertex set V, parameter $\delta > 0$

Find: function $f \in \mathbb{R}^V$

$$\max \langle f, Gf \rangle$$

$$f^2 = f$$

$$\mathbf{E} f \leq \delta$$



$f = \mathbb{1}_S$

Hypercontractivity implies SSE

$P =$ projector into span of eigenfunctions of G with eigenvalue $\geq \lambda$

Suppose $\|P\|_{2\to4} \ll 1/\delta^{1/4}$ and f is an optimal SSE solution.

Since $\|f\|_4/\|f\|_2 \geq \delta^{-1/4} \gg \|P\|_{2\to4}$, function f is *far* from image(P)

Hence, $\langle f, Gf \rangle \leq \big(\lambda + o(1)\big)\|f\|_2^2 \approx \lambda \cdot \delta$

G $=$ long-code graph $\mathrm{Cay}(\mathbb{F}_2^m, T)$ where $T = \{$points with Hamming weight $\varepsilon m\}$

$P =$ projector into span of eigenfunctions of G with eigenvalue $\geq \lambda = 0.1$

SoS proof of hypercontractivity:

$2^{O(1/\varepsilon)}\|f\|_2^4 - \|Pf\|_4^4$ is a sum of squares

G = long-code graph $\mathrm{Cay}(\mathbb{F}_2^m, T)$ where T = {points with Hamming weight εm}

P = projector into span of eigenfunctions of G with eigenvalue $\geq \lambda = 0.1$

SoS proof of hypercontractivity:

difference is sum of squares

$$\|Pf\|_4^4 \preccurlyeq 2^{O(1/\varepsilon)} \|f\|_2^4$$

For long-code graph, P projects into *Fourier polynomials* with degree $O(1/\varepsilon)$

Stronger ind. Hyp.:

$$\mathbf{E}\, f^2 g^2 \preccurlyeq 3^{d+e} \mathbf{E} f^2 \cdot \mathbf{E} g^2$$

where f is a generic degree-d Fourier polynomial
and g is a generic degree-e Fourier polynomial

$$\mathbf{E} f^2 = \sum_{S, |S| \leq d} \hat{f}_S^2$$

G = long-code graph $\mathrm{Cay}(\mathbb{F}_2^m, T)$ where T = {points with Hamming weight εm}

P = projector into span of eigenfunctions of G with eigenvalue $\geq \lambda = 0.1$

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Write $f = f_0 + x_1 \cdot f_1$ and $g = g_0 + x_1 \cdot g_1$ (degrees of f_1, g_1 smaller than d, e)

$$\begin{aligned}\mathbf{E}\, f^2 g^2 &= \mathbf{E}\, f_0^2 g_0^2 + \mathbf{E}\, f_1^2 g_0^2 + \mathbf{E}\, f_0^2 g_1^2 + \mathbf{E}\, f_1^2 g_1^2 + 4\mathbf{E}\, f_0 f_1 g_0 g_1 \\ &\preccurlyeq \quad \ldots \quad + 2\mathbf{E}\, f_0^2 g_1^2 + 2\mathbf{E}\, f_1^2 g_0^2 \\ &\preccurlyeq 3^{d+e}(\mathbf{E} f_0^2 + \mathbf{E} f_1^2) \cdot (\mathbf{E} g_0^2 + \mathbf{E} g_1^2) \quad \text{(ind. hyp.)}\end{aligned}$$

■

Let P be projector into d dimensional subspace of functions $f : V \to \mathbb{R}$

In time $\exp O(n^{2/q})$, can distinguish $\|P\|_{2\to q} = O(1)$ and $\|P\|_{2\to q} \gg 1$

Algorithm

Enumerate subspace if dimension $< O(n^{2/q})$

Otherwise, project standard basis vectors into the subspace and pick best

Analysis

worst case: $\|P\mathbb{1}_i\|_\infty = \frac{d}{n}$ and $\|P\mathbb{1}_i\|_2 = \frac{\sqrt{d}}{n}$ for all $i \in V$

$$\operatorname{Tr} P = d$$

$$\operatorname{Tr} P = \sum_i (P\mathbb{1}_i)_i \le \sum_i \|P\mathbb{1}_i\|_\infty$$

$$\operatorname{Tr} P = n \cdot \sum_i \|P\mathbb{1}_i\|_2^2$$

Finally, use $\|P\mathbb{1}_i\|_q \ge \|P\mathbb{1}_i\|_\infty / n^q$

■

Summary

Level-8 of SoS hierarchy refutes all known UG instances

show soundness via SoS proof

New connections between hypercontractivity & small-set expansion

and between ... & quantum separability

Open Problems

New UG instances from 2-to-4 norm hardness?

Stronger hardness for 2-to-4 norms?

Show that level-8 of SoS hierarchy solves *all* UG instances!

Thank you!